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ON PROBLEMS OF ESTIMATION FOR TWO PARAMETER

DECREASING FAILURE RATE DISTRIBUTIONS

APPLIED TO RELIABILITY

BY

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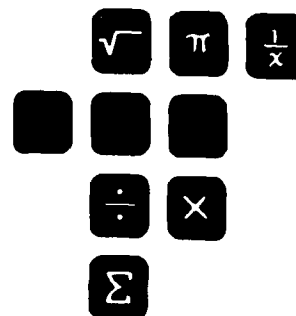ELECTE
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Problems of parametric estimation for samples from distributions which have decreasing rate are discussed. Some of these distributions are obtained by mixing different exponential populations, others represent mechanisms analogous to "work hardening" for each component where "old is better than new". Sufficient conditions are obtained that maximum likelihood estimations of appropriately chosen shapes and scale parameters exist in both cases.

0. ABSTRACT

In this paper some of the problems of parametric estimation for samples from distributions, which have decreasing failure rate, are discussed. Some of these distributions are obtained by mixing different exponential populations, others represent mechanisms analogous to ⁴work hardening⁸for each component where ⁸old is better than new⁸. Sufficient conditions are obtained that maximum likelihood estimators of appropriately chosen shape and scale parameters exist in both cases. The sample data which are available, when a decreasing failure rate distribution governs life, are usually censored with only a few failures observed; this limitation is dealt with. Actual data, obtained from the testing of integrated circuit electronic packages, are provided to illustrate these concepts and verify the applicability and usefulness of the techniques described.

Key Words

Reliability
Decreasing Failure Rate
Mixed Exponential
Censored Sample
Maximum likelihood Estimation
Burn-in

1. INTRODUCTION

The exponential distribution is currently one of the most widely used models in the study of life lengths, not only because of its early exposition, by Epstein and Sobel (1953), and the consequent development of the associated tests for, and estimators of, its parameters but also because of its intuitively understood properties, and its theoretical simplicity.

The exponential family of distributions, serves as the boundary between two important classes of life length distributions, namely, monotone increasing or monotone decreasing failure rates. These IFR and DFR families respectively, with their distributional properties, have played central roles in the development of reliability theory; see the text of Barlow and Proschan (1975).

If each component has a life distribution with constant failure rate, the value of which depends upon both the variability in its manufacture and the rigor of its service environment, then life data collected and pooled will then exhibit an apparent decreasing failure rate. The reliability of any such component at a time $t > 0$ is given by

$$R(t) = \int_0^\infty e^{-\lambda t} dG(\lambda), \qquad (1.1)$$

where G is the mixing distribution governing the failure rate variability.

It is known not only that the corresponding hazard function $-\ell n R(t)$ will be concave increasing for any mixing distribution G, see Barlow and Proschan p. 101, (1975), since the early paper of Proschan (1963), that there

are instances when such mixing occurs naturally and therefore distributions
with decreasing failure rates are of practical interest. Subsequently,
several specific models of DFR distributions were discussed by Cozzolino
(1968). But parametric families of DFR distributions have received little
attention compared with their IFR counterparts.

One model, which is of interest in its own right, is based on a gamma
mixture. If for some $\alpha, \beta > 0$ we take the density

$$G'(\lambda) = \frac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \qquad \text{for } \lambda > 0,$$

it then follows from equation (1.1) that the reliability function is

$$R(t) = (1+t\beta)^{-\alpha} \qquad \text{for } t > 0, \qquad (1.2)$$

with cumulative hazard, $Q = -\ell n R$,

$$Q(t) = \alpha \ell n (1+t\beta) , \qquad \text{for } t > 0, \qquad (1.3)$$

and failure rate $q(t) = \alpha\beta/ (1+t\beta)$. The density is given by

$$f(t) = -R'(t) = \frac{\alpha\beta}{(1+t\beta)^{\alpha+1}} \qquad \text{for } t > 0 . \qquad (1.4)$$

We see by direct integration, or conditional expectation, that

$$E[T] = \int_0^\infty R(t)dt \qquad = \qquad \begin{cases} \dfrac{1}{\beta(\alpha-1)} & \text{for } \alpha > 1 \\[2mm] \infty & \text{for } 0 < \alpha \le 1, \end{cases}$$

$$E[T^2] = \frac{\alpha}{\beta^2}\int_0^\infty \frac{x^2 dx}{(1+x)^{\alpha+1}} \quad = \quad \begin{cases} \dfrac{2}{\beta^2(\alpha-2)(\alpha-1)} & \text{for } \alpha > 2 \\[2mm] \infty & \text{for } 0 < \alpha \le 2, \end{cases}$$

and for $\alpha > 2$

$$Var(T) = \frac{\alpha}{\beta^2(\alpha-1)^2(\alpha-2)} \ . \tag{1.5}$$

These formulae have been given previously by Harris and Singpurwalla (1968) and Barlow and Proschan (1975).

This two-parameter family seems to have been introduced in this country by Lomax (1954) who regarded it as a generalization of a Pareto distribution, which is useful in business analysis.

Kulldorff and Vännman (1973) and Vännman (1976) have studied a best linear unbiased estimate for the parameters of this gamma mixed exponential model. They obtained a best linear unbiased estimate of the scale parameter β assuming that the shape parameter, α, was known and in a region restricted so that both the mean and the variance exist, namely $\alpha > 2$. When this restriction of $\alpha > 2$ cannot be met an estimate based on a few order statistics, which are optimally spaced, is claimed to be asymptotically best linear unbiased estimate, and tables of the weights as functions of the number of spacings are provided. In all cases, the shape parameter was assumed known and the sample was either complete or censored at the kth order observation. It is contended that BLUE estimates of the shape parameter are not attainable.

Harris and Singpurwalla (1968) examined the method of moments as an estimation procedure for this same model but again with the shape parameter restricted to $\alpha > 2$, and only for complete samples.

Harris and Singpurwalla (1969) also exhibited the maximum likelihood equations for complete samples for this model but without any discussion of computational methods for their solution,which was only implicitly defined. In Kulldorff and Vännman (1973) there is a brief bibliography of results on parametric estimation for the Lomax distribution, with density given by (1.4), under various assumptions.

As a consequence of the widespread adoption of integrated circuitry, at present,life testing in electronic manufacturing virtually always provides incomplete samples. This is because tests are censored, because of the expense, and the paucity of failures due to the high reliability of integrated circuit devices. Such service life data cannot be adequately treated by any of the presently known statistical techniques without employing Bayesian arguments, with their utilization of subjective information. This would indicate the need for an objective estimation procedure, making use of the only type of data available, and without the potential for bias inherent in Bayesian priors.

In this paper the maximum likelihood estimates are obtained for both the shape and scale parameters, from a class of two parameter families (including the Lomax distribution), and necessary and sufficient conditions are given for their existence. These estimates are derived for censored data and *a fortiori* for complete samples, even with a paucity of failure observations.

The conditions for the existence of the maximum likelihood estimates apply even to the case where the mean and variance of the life distributions might not exist, e.g., in the case of a Lomax distribution when $0 < \alpha < 2$. Moreover, the estimates of the shape parameter α obtained from actual data indicate it is important to have estimation procedures valid in this range.

2. DECREASING FAILURE RATE MODELS

We now postulate the underlying physical processes which determine the length of life under consideration. Firstly let us suppose the quality of construction determines a level of resistance to stress which the component can tolerate. The service environment provides shocks of varying magnitude to the component and failure takes place when, for the first time, the stress from an environmentally induced shock exceeds the strength of the component.

If the time between shocks of any magnitude is exponentially distributed, with a mean depending upon that magnitude, then the life length of each component will be exponentially distributed with a failure rate which is determined by the quality of assembly. It follows that *each component in service will have a constant failure rate* but that the variation in manufacturing and inspection procedures causes some components to be extremely good, with a low failure rate, while a few others are bad, with a high failure rate, and most are in-between.

Alternatively, let us consider structures subjected to dynamic stresses of such a nature that the first stresses to which they are subjected if not severe enough to initiate failure only cause localized yielding and deformation thus affecting local stress relief and reinforcement. Such behavior increases their ability to withstand future stresses and could be thought of as "work hardening". In such cases older structures in service actually have greater resistance to fatal shocks than younger ones; i.e., *each component in service has a decreasing failure rate*. An analogous behavior exists for increased strength or immunity in biological systems.

This may be thought of as "the older the better" at least for certain periods and purposes. Usually the failure mechanism and its interpretation in these cases is quite different from that of a mixed exponential model.

Let λ be the measure of the lack of resistance to shock for a component, the life of which, say X_λ, will be exponential. If the variability of manufacture determines the frequency of the different λ-values, which we describe by a r.v., say Λ, with distribution G, then $T = X_\Lambda$ is the life length of a component selected at random from those manufactured; it will have a survival distribution which can be written as the conditional expectation

$$E_\Lambda P[X_\Lambda > t \mid \Lambda] = \int_0^\infty e^{-\lambda t} dG(\lambda) \ .$$

We restrict our attention to mixing distributions G with unit mean the choice of which determines the distribution of life.

For the random life length T having reliability R we postulate a cumulative hazard, for some known $0 < \gamma \le 1$, of the form

$$-\ell n R(t) = \alpha \int_0^{t\beta} [q(x)]^\gamma \, dx$$

where $\alpha > 0$, $\beta > 0$ are the unknown parameters for which, without loss of generality, the scale of λ may be chosen so that the mixed exponential hazard

$$q(t) = \frac{\int_0^\infty \lambda e^{-\lambda t} dG(\lambda)}{\int_0^\infty e^{-\lambda t} dG(\lambda)} \tag{2.1}$$

is *standardized* i.e. $q(0) = \int_0^\infty \lambda \, dG(\lambda) = 1$.

When q and γ have been specified we write $T \sim J(\alpha,\beta;q,\gamma)$ or $T \sim J(\alpha,\beta)$ and we call this the *Afanas'ev generalization* model for life lengths with unknown parameters α,β.

It is clear however that this parameterization, with $\gamma \neq 1$, may alter a mixed-exponential distribution to one which is not mixed-exponential but merely a decreasing failure rate distribution.

The gamma mixture of exponentials has been previously introduced in equation (1.2). Let us now consider its Afanas'ev generalization. For a given $0 < \gamma \leq 1$ we have for its standardized hazard rate q ,

$$q_1(t) = (1 + t)^{-\gamma} \quad \text{for} \quad t > 0 , \tag{2.2}$$

from which we find the corresponding hazard function to be

$$Q_1(t) = \begin{cases} [(1+t)^{1-\gamma}-1]/(1-\gamma) & \text{for} \quad 0 < \gamma < 1 \\ \ln(1+t) & \text{for} \quad \gamma = 1 . \end{cases} \tag{2.3}$$

For $\gamma = 1$, this is a Lomax distribution with shape parameter of unity. It is otherwise a Weibull distribution with a location parameter of minus one and a shape parameter of less than unity. In the Soviet Union it was introduced by Afanas'ev (1940) as a distribution for fatigue life.

We now examine the distribution which results from the mixing of two exponential distributions. It is presumed that owing to occasional laxity in quality control there is a low probability p of passing a component containing a defect which can cause a high failure rate $\lambda > 1$. But there is a high probability of passing a component having the nominal (low) failure rate,

which without loss of generality we take to be unity.

The reliability of this Bernoulli-mixed exponential population is

$$R(x) = pe^{-\lambda x} + (1-p)e^{-x} \text{ for } x > 0 .$$

Whence, we find the standardized hazard to be, for $x > 0$

$$Q_2(x) = -\ell np + \frac{\lambda x}{\mu} - \ell n(1 + re^{\nu x}) \tag{2.4}$$

with

$$r = \frac{1-p}{p}, \quad \nu = \frac{\lambda-1}{\mu} > 0, \quad \mu = p(\lambda+r) . \tag{2.5}$$

We shall write, respectively,

$$T \sim J_1(\alpha,\beta;\gamma) \quad \text{or} \quad T \sim J_2(\alpha,\beta:p,\lambda)$$

whenever the standardized hazard R_1 , is defined in (2.3) or Q_2 as defined in (2.4). Also without further mention we shall use the same sub-script to denote other functions associated with these cases.

We now comment that it is possible to introduce an alternative parameter-ization to that of (2.1), namely

$$R(t) = e^{-\frac{\alpha}{\beta} Q(t\beta)} \quad \text{for } t > 0; \alpha,\beta > 0, \tag{2.6}$$

where Q is a known cumulative hazard. The advantage is that in this form we see clearly that $\beta \to 0$ implies $R(t) \to e^{-\alpha t}$. Thus the limiting case is exponential life. In either case one sees that β measures the degree of departure from a constant failure rate in terms of Q and determines the rate of improvement in expected life with use. This, in a sense, is a reliability growth. The disadvantage is that β in (2.6) is now no longer a scale parameter and the likelihood equations, to be obtained subsequently, seem

somewhat more complicated. However, the transformation $(\alpha,\beta) \rightarrow (\alpha/\beta,\beta)$ is a 1-1 mapping of the positive quadrant into itself so that any maximum likelihood estimates for one parameterization could be immediately transformed to the other.

3. A CLASS OF TWO PARAMETER FAMILIES OF MIXED EXPONENTIALS

We shall postulate a class $\mathcal{D}$ of concave hazard functions (or decreasing hazard rate functions correspondingly).

Each element of this class will generate a two parameter family, which is a subset of the DFR distributions. Reliability will be of the form, where $Q \in \mathcal{D}$ is a given hazard function,

$$R(t) = e^{-\alpha Q(t\beta)} \qquad t > 0; \; \alpha, \beta > 0,$$

here $q = Q'$ is a decreasing hazard rate which is twice differentiable, standardized, i.e. $q(0) = 1$, and satisfies

1° $\psi \equiv xq$ is increasing,

2° q is log-convex,

3° $\zeta = 1 + xq'/q$ is non-negative, does not exceed unity and tends to a limit in $[0,1]$ as $x \to \infty$.

The question arises, "Where did such assumptions come from and what distributions, if any, satisfy them?" One sees immediately that if $q \in \mathcal{D}$ then $q^\gamma \in \mathcal{D}$ for any $\gamma \in (0,1]$ so that $\mathcal{D}$ is closed under fractional powers of the failure rate. The Afanasév generalizations of the Lomax distribution has failure rate q_1 given by (2.2). One sees that q_1 is decreasing and $\ln q_1$ is convex. Moreover, one checks easily that ψ_1 is increasing and in this case

$$\zeta_1(t) = [1 + (1-\gamma)t]/(1+t) \tag{3.2}$$

is decreasing with $\zeta_1(\infty) = 1-\gamma$. So in this archtypical example assumptions 1°, 2°, and 3° are met.

In the Bernoulli mixture of the two exponentials we find from (2.4)

$$q_2(x) = \frac{\lambda}{\mu} - \frac{r\nu}{(e^{-\nu x}+r)} \ , \quad q_2'(x) = \frac{-\nu^2 re^{-\nu x}}{(e^{-\nu x}+r)^2} \tag{3.3}$$

and since $r > 1$,

$$q_2''(x) = \nu^3 re^{-\nu x}(r-e^{-\nu x})/ \ (e^{-\nu x}+r)^3 \ > 0 \ , \tag{3.4}$$

thus

$$\zeta_2(x) = 1 - \frac{xr\nu^2 e^{-\nu x}}{(\lambda e^{-\nu x}+r)(e^{-\nu x}+r)}$$

so that we see $\zeta_2(\infty) = 1$.

We next show that ψ_2 is an increasing function. To see this we note that $\psi_2'(x) \geq 0$ for all $x > 0$ iff, after simplification,

$$\lambda e^{-\nu x} + r^2 e^{\nu x} \geq \nu r\mu x - r(1+\lambda), \quad \text{for all } x > 0.$$

Let $y = e^{\nu x}$ then the inequality above becomes clearly true for $1 \leq y \leq y_0$, where $\ell n \ y_0 = (1+\lambda)/(1-\lambda)$. Let $y = y_0 t$ for $t > 1$, then the inequality to be proved becomes

$$\frac{\lambda}{y_0 t^2} + r^2 y_0 \geq \nu r\mu \frac{\ell n \ t}{t}, \text{for all } t > 1.$$

But the right hand side is maximized at $t = e$, so it is sufficient to have $y_0 \geq \nu\mu/re$, which is implied by

$$p \leq \frac{1}{1 + (\lambda-1)e^{2/(\lambda-1)}} \simeq \frac{\lambda-1}{\lambda(\lambda+1)} \ .$$

We now must show that $\ln q$ is convex. To prove this it is sufficient to show that $q_2 q_2'' \geq (q_2')^2$. But substitution and simplification from (3.3) and (3.4) into the equation above shows that this inequality is true for all $x > 0$ iff $r \geq \sqrt{\lambda}$, which is equivalent with $p \leq (1 + \sqrt{\lambda})^{-1}$. We spare the reader further details.

We claim that these inequalities are virtually always true in practice since if, for example, λ should be as high as 10, then the probability of passing such a bad component, with a failure rate ten times the nominal design rate, must not exceed .082. This would seem to be a reasonable assumption, at least for firms that intend to remain in business. Furthermore this demonstration shows that not all decreasing failure rate distributions satisfy our assumptions.

Of course not all DFR distributions can be obtained by an exponential mixture such as given in (1.1). For an exact description of the extreme points of the class of DFR distributions, see Langberg et al [9]. Moreover only particular exponential mixing distributions thought to be of practical interest will concern us here, along with their corresponding Afanas'ev generalizations. Thus we will examine only a subclass of the DFR distributions.

We now ask what are sufficient conditions on the mixing distribution so that assumptions 1°, 2° and 3° will be satisfied for any Afanas'ev generalizations of mixed exponential? In the following discussion we adopt a notation omitting the limits of integration when they extend from 0 to ∞.

Theorem 1: If the mixing distribution G is such that the function

$$K(x,y) = G(x/y) \quad \text{for} \quad x,y > 0$$

is totally positive of order 2, then 1° is satisfied.

Proof: Making a change of variable in the definition we see

$$\psi(t) = \int y e^{-y} dG(y/t) \Big/ \int e^{-y} dG(y/t).$$

Upon integrating numerator and denominator by parts, we obtain

$$\psi(t) + 1 = \frac{\int G(y/t) y e^{-y} dy}{\int G(y/t) e^{-y} dy} \quad .$$

Let $t_1 > t_2$, we must show $\psi(t_1) \geq \psi(t_2)$. This is true iff

$$0 \leq \begin{vmatrix} \int G(y/t_1) y e^{-y} dy, & \int G(y/t_1) e^{-y} dy \\ \int G(y/t_2) y e^{-y} dy, & \int G(y/t_2) e^{-y} dy \end{vmatrix} \quad .$$

By applying the basic composition formula of Karlin, see e.g., p. 100, Barlow and Proschan (1975), to the right-hand side above, it becomes equal to

$$\iint_{y_1 < y_2} \begin{vmatrix} G(t_1/y_1), & G(t_1/y_2) \\ G(t_2/y_1), & G(t_2/y_2) \end{vmatrix} \times \begin{vmatrix} y_1 e^{-y_1}, & e^{-y_1} \\ y_2 e^{-y_2}, & e^{-y_2} \end{vmatrix} dy_1 \, dy_2$$

Clearly the second determinant is negative and that the first is negative, by definition of TP-2, can be seen by setting $x_1 = t_2$, $x_2 = t_1$. QED.

The hypothesis of this theorem has a relation with Polya-Frequency functions of order 2, if it were expressed as the difference of the logarithms rather than as a ratio.

We now have the important second

Theorem 2: For any G-mixed exponential distribution, the failure rate q is a log-convex and 2° is satisfied.

Proof: From equation (2.1) we find

$$(\ell nq)'(t) = q(t) - \frac{\int \lambda^2 e^{-\lambda t} dG(\lambda)}{\int \lambda e^{-\lambda t} dG(\lambda)} \tag{3.6}$$

and hence

$$(\ell nq)''(t) = \frac{\int \lambda^2 e^{-\lambda t} dG(\lambda)}{\int e^{-\lambda t} dG(\lambda)} - \left[\frac{\int \lambda e^{-\lambda t} dG(\lambda)}{\int e^{-\lambda t} dG(\lambda)}\right]^2$$

$$+ \frac{\int \lambda^3 e^{-\lambda t} dG(\lambda)}{\int \lambda e^{-\lambda t} dG(\lambda)} - \left[\frac{\int \lambda^2 e^{-\lambda t} dG(\lambda)}{\int \lambda e^{-\lambda t} dG(\lambda)}\right]^2$$

It is sufficient to show that $(\ell nq)'' \geq 0$. To see the first difference is positive we fix t and set

$$dH(\lambda) = \frac{e^{-\lambda t} dG(\lambda)}{\int e^{-\lambda t} dG(\lambda)} . \tag{3.7}$$

H then becomes a distribution function and the first difference can be written

$$\int \lambda^2 dH(\lambda) - [\int \lambda dH(\lambda)]^2 ,$$

which is a variance and is positive. The second difference is handled similarly. QED.

We now examine the behavior of ζ for a mixed exponential, in particular its asymptotic behavior determined by the behavior of the mixing distribution G near 0.

We do not consider as reasonable for our application mixing distributions which are of infinite order at the origin, such as $G(x) = e^{-x^{-2}}$ for $x > 0$. We admit for consideration only distributions G which are

(I) discrete in some neighborhood of zero, or

(II) there exists $\kappa > 0$ such that $x^{-\kappa}G(x) \to c > 0$ as $x \to 0$.

Theorem 3. For any G-mixed exponential distribution, $\zeta(0) = 1$, and ζ is

initially decreasing. Since limit exists we have $\zeta(\infty) \in [0,1]$ and:

If G is of type (i) then $\zeta(\infty) = 1$

If G is of type (ii) then $\zeta(\infty) = 0$.

Proof: Let us consider

$$1 - \zeta(t) = \frac{-t\, q'(t)}{q(t)} = \frac{\int x^2 e^{-x} dG(x/t)}{\int x e^{-x}\, dG(x/t)} - \frac{\int x e^{-x} dG(x/t)}{\int e^{-x} dG(x/t)},$$

which we see can be regarded as a variance divided by a mean and so is positive always.

We see ζ initially decreases linearly with slope $q'(0)$ since

$$\zeta'(0) = \lim_{x \to 0} \frac{\zeta(x)-1}{x} = q'(0).$$

We now consider the limit as $t \to \infty$. Integrating by parts, above, we obtain

$$\frac{\int (x/t)^{-\kappa} G(x/t)\, x^{\kappa} d(x^2 e^{-x})}{\int (x/t)^{-\kappa} G(x/t)\, x^{\kappa} d(x e^{x})} - \frac{\int (x/t)^{-\kappa} G(x/t)\, x^{\kappa} d(x e^{-x})}{\int (x/t)^{-\kappa} G(x/t) x^{\kappa} de^{-x}}$$

$$\to \quad \frac{\Gamma(\kappa+2)}{\Gamma(\kappa+1)} - \frac{\Gamma(\kappa+1)}{\Gamma(\kappa)} = 1 \qquad \text{as } t \to \infty.$$

The alternative when G is discrete is verified by a Tauberian argument which is similar. QED.

Comment: The exact conditions on the mixing distribution G which insure that $1 - \zeta$ is unimodal are not easily seen in general. Such behavior can be easily checked in the instances given.

<u>Theorem 4</u>. For any Afanas'ev generalization of a mixed exponential, say $q = q_0^\gamma$ for some $\gamma \in (0,1]$ where the standardized q_0 is given by (2.11) with the mixing distribution of type (i) or (ii) then there exists a $\nu \in [0,1]$ such that $x^\nu q(x) \to a > 0$ as $x \to \infty$.

Proof:

Let us first suppose that $\nu = 1$ so that q is a mixed exponential hazard rate. We claim that $\nu = 0$ or 1 suffices accordingly as G is of type (i) or of type (ii).

If (i) then for some $0 < \lambda_0 < \epsilon$, $p_0 > 0$

$$R(t) = p_0 e^{-\lambda_0 t} + \int e^{-\lambda t} dG_\epsilon(\lambda)$$

with the obvious interpretation of G_ϵ with support inside (ϵ, ∞), in which case

$$q(t) = \frac{p_0 \lambda_0 + \int \lambda e^{-(\lambda - \lambda_0)t} dG_\epsilon(\lambda)}{p_0 + \int e^{-(\lambda - \lambda_0)t} dG_\epsilon(\lambda)} \to \lambda_0 \neq 0$$

as $t \to \infty$. Thus $\nu = 0$ suffices.

If (ii) then for some $\kappa > 0$, $x^{-\kappa} G(x) \to c \neq 0$ as $x \to 0$ and from (3.5), after integrating by parts, we have

$$\psi(x) = - \frac{\int G(t/x) \, d(t \, e^{-t})}{\int G(t/x) \, d e^{-t}} \, .$$

Again multiplying numerator and denominator $x^{-\kappa}$ we have as $x \to \infty$

$$\psi(x) = \frac{\int (x/t)^\kappa \, G(t/x) \, t^k \, d(t \, e^{-t})}{\int (x/t)^\kappa \, G(t/x) \, t^k \, d e^{-t}} = \kappa \, .$$

Thus $\nu = 1$ suffices.

We next consider a failure rate which is a fractional power of a mixed exponential for which $x^\delta q_0 \to c \neq 0$ for $\delta = 0$ or 1. Then $\nu = \delta \gamma$ will suffice since

$$x^\nu q_0^\gamma = (x^\delta q_0)^\gamma \to a > 0 \quad \text{as} \quad x \to \infty . \quad \text{QED.}$$

In any life length model one is interested in the distribution resulting when independent components having such a distribution are put into a series system. From the concavity of the hazard follows the

Remark: If components with independent life lengths $T_i \sim J(\alpha_i, \beta_i)$ $i = 1,\ldots,M$ are in series then the life of the system T will be in the same DFR class and satisfies the stochastic inequality

$$T = \min T_i \gtrsim J(\Sigma \alpha_i, \Sigma \alpha_i \beta_i / \Sigma \alpha_i)$$

with equality when $\beta_i \equiv \beta$.

Another property, which one would ask of any DFR model, is what improvement can be made by "burning-in" a component with such a life? In practice it is often assumed that as a result of a burn-in period, surviving components are exponentially lived. In fact burn-in tests are often required in certain electronic component manufacturing with a statement of the ultimate failure rate so obtained.

Of course, not all decreasing failure rate distributions do become constant after some finite initial period, but that is a model which is often thought to be appropriate. This indicates the importance of the second model introduced and of its utility in a determination of the economic value of the stochastically extended life.

The residual life T_τ of a component with new life length T and a burn-in of duration $\tau = 0$ is the conditional life remaining after time τ given that it is alive then; i.e. $T_\tau = [T - \tau \mid T > \tau]$.

The residual life of any G-mixed exponential is again a mixed exponential but with a different mixing distribution. The residual life will have density $f(y) = ce^{-\tau y} dG(Y)$, where c is the normalizing constant.

It is easily seen that $T_\tau \overset{st}{>} T$ for all $\tau > 0$ as long as τ is within the initial interval on which q is strictly decreasing. Usually the burn-in time τ is increased until any incremental decrease in the residual failure rate is not worth the increment cost of testing. This point will necessarily be somewhat different for each particular Q .

Remark: A burn-in of τ units of time on a component with a new life $T \sim J_1(\alpha, \beta)$ will yield a residual life

$$T_\tau \sim J_1[\alpha(1+\tau\beta)^{1-\gamma}, \beta/(1+\tau\beta)].$$

If $T \sim J_2(\alpha,\beta:p,\lambda)$ then a burn-in length τ will only alter the proportion of high failure rates, viz.,

$$T_\tau \sim J_2(\alpha,\beta:p',\lambda)$$

where the altered proportion is given

$$p' = pe^{-\lambda\tau\beta}/(pe^{-\lambda\tau\beta}+(1-p)e^{-\tau\beta}).$$

4. ESTIMATION OF PARAMETERS USING INCOMPLETE SAMPLES

The samples, that are obtained when components having a DFR life distribution are tested, are virtually always incomplete in the sense that testing is stopped before all components have failed. A datum on a component that "failure has not yet occurred after a specified life" is called an alive time or a run-out. Samples containing such observations are *censored*. Samples in which life tests are truncated at some preassigned ordered observation occur infrequently, in our experience, when electronic components are tested.

It is assumed throughout this section that we are given a sample $\underline{t} = (t_1,\ldots,t_k,\ldots,t_n)$ where $t_1,\ldots,t_k$ are ordered observations of times of failure while $t_{k+1},\ldots,t_n$ are the ordered observed alive-times, with $1 \le k \le n$. All observations are obtained from testing components having a $J(\alpha,\beta)$ distribution with $Q \in \mathfrak{Q}$ and γ specified but with unknown parameters α and β.

We now introduce notation for the two empirical distributions respectively, call them F_k and F_n. We set

$$F_k(y) = \frac{1}{k} \{\# \text{ of } t_i \le y \text{ for } i = 1,\ldots,k\}$$

and similarly for F_n, and we make the notational convention, to be used subsequently for any function g, that

$$\overline{g}(x) = \int_0^\infty g(xt)dF_n(t), \quad \tilde{g}(x) = \int_0^\infty g(xt)dF_k(t).$$

Some results will now be given on maximum likelihood estimation of the unknown shape and scale parameters in the case of censored samples with $Q \in \mathfrak{Q}$ and $\gamma \in (0,1]$ specified.

Remark 5: When the scale parameter $\beta > 0$ is known, there exists a unique m.l.e. of α, say $\hat{\alpha}$, given explicitly by

$$\hat{\alpha} = k/n\bar{Q}(\beta). \tag{4.1}$$

This result is not new. If β is known and Q given, concave or not, then the values $y_i = Q(t_i\beta)$ for $i = 1,\ldots,n$ can be calculated. They are the alive and dead times from an exponential distribution with unknown failure rate α. The total life statistic divided by the number of failure yields the usual maximum likelihood estimate of the mean life.

What is obviously true is that not any set of n positive numbers, say $\underline{t} = (t_1,\ldots,t_k,\ldots,t_n)$ with $k \le n$ designated as failure times and the remainder as alive times, can be used to estimate both unknown parameters for any $Q \in \mathfrak{Q}$. In some sense the sample must be close to what would be likely from such a distribution. We now have for a specified $Q \in \mathfrak{Q}$ the result

Theorem 6: For given $\alpha,\underline{t}$ there exists a unique m.l.e. of β, denoted by $\hat{\beta}$, and it is given implicitly as the (positive) root of the equation

$$\tilde{\zeta}(x) - \frac{\alpha n}{k}\bar{\psi}(x) = 0 \tag{4.2}$$

whenever

$$\bar{\psi}(1/\delta_0) \ge \frac{k}{\alpha n} \tag{4.3}$$

where ζ is defined in (3.1) and x_0 is the maximum value such that $\bar{\zeta}$ is decreasing n $(0,x)$ and where $\delta_0 = \sup\{\tilde{\zeta}(x)/x: x \ge x_0\}$.

We remark that (4.3) is always true if ζ is decreasing. One notes from (3.2) that ζ_1 is decreasing with $\zeta_1(\infty) = 1 - \gamma$, but ζ_2 is initially convex decreasing then ultimately concave increasing with $\zeta_2(\infty) = 1$.

For completeness we state theorems 6, 7 and 8 together with comments and explanation. The proofs are deferred until later.

The failure to fulfill the condition (4.3) for a sample would indicate that the presupposed choice of α or q should be reexamined. That is to say, either the initial failure rate α is not large enough to reflect accurately the number of failures observed or the induced function ζ does not decrease over a sufficiently long interval.

Some simpler sufficient conditions to check are as follows.

<u>Theorem 7</u>: For given $\alpha, \underline{t}$, Q ϵ $\mathfrak{Q}$ there exists a unique m.l.e. $\hat{\beta}$ of β which is given as the solution of (3.6), if ζ decreases on $(0, t_0)$ and

$$q(t_m t_0/t_k) \geq K \equiv kt_k/\alpha n\overline{t}\, t_0 \; .$$

Here t_k is the maximum dead time and t_m the maximum of t_k and t_n.

Theorems 6 and 7 say that if α is known then a m.l.e. of β exists and, usually, is easily computed. Otherwise the asymptotic behavior of ζ, determined from q, must be accounted for.

We now repeat this discussion when both α and β are unknown. It is shown that both m.l.e.'s exist whenever the sample satisfies the condition $2\overline{\overline{t}\,t} < \overline{t^2}$. If the sample fails to satisfy this condition then the model may not be appropriate; either and a constant failure rate model or a convex failure rate model may be indicated rather than a decreasing failure rate .

<u>Theorem 8</u>: For a given sample $\underline{t}$, with Q ϵ $\mathfrak{Q}$ specified and α, β both unknown, the m.l.e. of β, say $\hat{\beta}$, exists as the smallest positive root of the equation

$$\tilde{\zeta}(x) - \Phi(x) = 0, \quad \text{for } \alpha > 0$$

where $\Phi = \overline{\psi}/\overline{Q}$, iff the sample satisfies the inequality

$$2\overline{t}\,\tilde{t} < \overline{t^2} \; . \tag{4.4}$$

When $\hat{\beta}$ has been determined, the m.l.e. of α, say $\hat{\alpha}$, is then given by an

analogue of equation (4.1), namely,

$$\hat{\alpha} = k/n\overline{Q}(\hat{\beta}).$$ (4.5)

We now give the proofs of the preceding theorems.
The interchange of differentiation and integration in what follows is justified easily, using the dominated convergence theorem, since the functions are bounded by unity.

<u>Proof</u> <u>of</u> <u>Theorem</u> <u>6</u>: As a result of the testing, assume that we have the vector $\underline{t} = (t_1, \ldots, t_k, \ldots t_n)$ corresponding to the observed events

$[T_i = t_i]$ for $i = 1, \ldots, k$ and $[T_i > t_i]$ for $i = k+1, \ldots, n$.

By definition the log-likelihood L is given by

$$e^L = \prod_{i=1}^{k} f(t_i) \prod_{i=k+1}^{n} R(t_j).$$

Substituting from (2.1) and taking logarithms we find

$$L = k\ln(\alpha\beta) + \sum_{1}^{k} \ln q(t_i\beta) - \alpha \sum_{1}^{n} Q(t_i\beta)$$

$$= k\ln(\alpha\beta) + \int_0^\infty \ln q(\beta t) dF_k(t) - \alpha n \int_0^\infty Q(\beta t) dF_n(t).$$

Dividing by the constant k, we write

$$L(\alpha,\beta|\underline{t}) = \ln\alpha + \ln\beta + \widetilde{\ln q}(\beta) - \frac{\alpha n}{k} \overline{Q}(\beta)$$ (4.6)

making use of the notational convention introduced earlier. Thus for given $\beta > 0$, the likelihood $L(\cdot|\beta,\underline{t})$ is concave on $(0,\infty)$ and the m.l.e. of α exists uniquely and can be obtained from the equation $L'(\alpha|\beta,\underline{t}) = 0$, the explicit solution of which is given in equation (4.1).

Since Q is concave increasing so is the average $\overline{Q}$. On the other hand ℓnq is convex, consequently so is $\widetilde{\ell nq}$. Thus for given $\alpha > 0$ the likelihood $L(\beta|\alpha,\underline{t})$ is the sum of a convex and concave function. It can be written, neglecting constants depending upon α or $\underline{t}$, as

$$L(\beta|\alpha,\underline{t}) = \widetilde{\ell n}\psi(\beta) - \frac{\alpha n}{k} \overline{Q}(\varepsilon).$$

If we define $A(\beta) = L'(\beta|\alpha,\underline{t})$ we see that we must examine the roots of

$$A(x) = \tilde{\zeta}(x) - \frac{\alpha n}{k} \overline{\psi}(x).$$

To examine the behavior of this difference A, we note since

$$\zeta = 1 + x(\ell nq)'$$

that

$$\tilde{\zeta}'(x) = \int_0^\infty t\zeta'(tx)dF_k(t) = \int_0^\infty [xt^2(\ell nq)''(xt) + t(\ell nq)'(xt)]dF_k(t).$$

By assumption $(\ell nq)'' > 0$ but $(\ell nq)' < 0$. Now we see by Theorem 4,

$$\lim_{x \downarrow 0} \tilde{\zeta}'(x) = \zeta'(0)\tilde{t} < 0.$$

Thus for any fixed t with x sufficiently near zero the function $\zeta'(tx)$ will be negative but for larger values of x the function may become positive. Thus we conclude that the sample function $\tilde{\zeta}$ must initially be decreasing and then could become oscillatory, but approaches $\zeta(\infty)$, in the unit interval. Note also that $\overline{\psi}$ is the mean of increasing functions and is increasing.

Recall that x_0 is the maximum value such that $\tilde{\zeta}$ is decreasing on $(0,x)$ and

$$\delta_0 = \max\{\tilde{\zeta}(x)/x : x > x_0\} ;$$

Thus $\tilde{\zeta}(x)$ never crosses the ray $\zeta_0 x$ for $x > x_0$. Since we have required that

$$\overline{\psi}(1/\delta_0) \geq \frac{k}{\alpha n}$$

there can have been only one value $\hat{\beta}$ such that $A(\hat{\beta}) = 0$ with $\hat{\beta} < x_0$. This follows since $\overline{\psi}(x)/x$ is a decreasing function and cannot have obtained the value $k/\alpha n$ for $x < \delta_0$. But $\overline{\psi}(x)$ cannot intercept $\tilde{\zeta}(x)$ for $x > x_0$ since $\overline{\psi}$ lies above the ray and $\tilde{\zeta}$ below it. But because $\overline{\psi}(0) = 0 < \tilde{\zeta}(0) = 1$ and $\overline{\psi}(x_0) > \tilde{\zeta}(x_0)$ with $\tilde{\zeta}$ decreasing and $\overline{\psi}$ increasing, there must have been exactly one intersection prior to x_0. This proves theorem 6. QED.

Proof of theorem 7. Since ζ decreases on $(0,t_0)$, it follows that $\tilde{\zeta}$ must decrease on $(0,x)$ where $x = t_0/t_k$. It is sufficient to show $\overline{\psi}(x) \geq k/\alpha n$, since there can have been only one intersection previously and none afterward. Now $q(t_m x) \geq K$ implies that

$$\overline{\psi}(x)/x = \int_0^\infty tq(tx)dF_n(t) \geq \overline{t}\,K$$

since $q(tx)$ is decreasing as a function of t and $q(t_i x) \geq q(t_m x)$ for $i = 1,\ldots,n$. Thus $\overline{\psi}(x) \geq x\overline{t}\,K = \frac{k}{\alpha n}$. This concludes the proof of theorem 7. QED.

Proof of theorem 8. Consider the likelihood function $L(\alpha,\beta|\underline{t})$ defined over the positive quadrant as given in (4.0). All the stationary points, which are determined by $\underline{t}$, can be found from the simultaneous solution of $\frac{\partial L}{\partial \alpha} = 0$, $\frac{\partial L}{\partial \alpha} = 0$. This yields the two equations in α and β;

$$\tilde{\zeta}(\beta) = \frac{\alpha n}{R}\,\overline{\psi}(\beta)\,;\quad \frac{1}{\alpha} = \frac{n}{R}\,\overline{Q}(\beta)\ .$$

Combining these into a single equation in the unknown β, we are led to seek $\hat{\beta}$ as a zero of the difference $\tilde{\zeta}(x) - \Phi(x)$ for $x > 0$ where $\Phi = \overline{\psi}/\overline{Q}$.

Consider the likelihood as given in (4.6) as a function of α, β into which we substitute the relation displayed above to obtain the likelihood maximized with respect to α for any value of β. Neglecting constants depending upon $\underline{t}$, this function can be written, now using argument x, as

$$L(x) = \widetilde{\ell nq}(x) - \ell n[\overline{Q}(x)/x].$$

Interchanging the order of integration in $\overline{Q}(x)/x$ we obtain

$$\overline{Q}(x)/x = \overline{t} \int_0^\infty q(tx) \, dW_n(t) \equiv \overline{t} \cdot q^*(x)$$

where W_n is the distribution, with density given by

$$W_n'(t) = [1 - F_n(t)]/\overline{t} \qquad \text{for } t > 0.$$

Again neglecting constants, we have

$$L(x) = \int_0^\infty \ell nq(tx) \, dF_k(t) - \ell n[\int_0^\infty q(tx) \, dW_n(t)$$

$$= \widetilde{\ell nq}(x) - \ell nq^*(x) . \tag{4.8}$$

Using a Maclaurin expansion of q we see

$$L(x) = x \, q'(0) \, [\tilde{t} - t^*] + 0(x^2) \qquad \text{as } x \to 0,$$

where

$$\tilde{t} = \int t \, dF_k(t) \quad \text{and} \quad t^* = \int t \, dW_n(t) = \overline{t^2}/2\overline{t} .$$

Thus L is positive in a neighborhood of zero iff $\tilde{t} < t^*$; and since $xL' = \tilde{\zeta} - \Phi$ this difference will be positive in a neighborhood of zero under the same condition.

We now examine the behavior of L under the more strict condition that the sample distributions F_k and W_n as defined satisfy $F_k \geq W_n$. It then follows that $\tilde{q} \geq q^*$ and since ℓn is concave, that $\ell n q^* \geq \overline{\ell n q}$. Hence from (4.8) we see that

$$\widetilde{\ell n q} - \ell n \tilde{q} \leq L \leq \widetilde{\ell n q} - \overline{\ell n q} .$$

Now one notices the lower bound is always negative while the upper bound is always positive. But near the origin L is approximately equal to the upper bound while for larger values of its argument L is approximately equal to the lower bound and for intermediate values L makes a smooth transition between.

5. THE COMPUTATION OF $\hat{\beta}$ FOR ARBITRARY SAMPLES.

As a matter of practical calculation we are concerned with the smallest root of the difference $\tilde{\zeta} - \phi$, in the case when ζ is monotone decreasing. Let us consider the composite function $f(x) = \tilde{\zeta}^{-1}[\phi(x)]$ in a neighborhood of zero with the location of the smallest crossing, if there is more than one, of the 45° line. These are equivalent.

An alternative expression for ϕ is $\phi = x\bar{Q}'/\bar{Q}$, with $\bar{Q}'$ a convex function decreasing between $\bar{t}$ and $\bar{t} \cdot q(\infty)$, while "the smoothed $\bar{Q}'$ ", viz., $\bar{Q}/x$, decreases between the same limits at a slower rate. It follows that $\bar{Q}/x \geq \bar{Q}' \geq 0$ so that $0 \leq \phi \leq 1$. The ϕ begins at unity, initially decreases at a decelerating rate and tends ultimately to $\zeta(\infty)$. To see this note

$$\phi(\infty) = \lim_{\infty} \bar{\psi}'/\bar{Q}' = \lim_{x \to \infty} \frac{\int t^{1-\nu}(xt)^{\nu}q(tx)\zeta(tx)dF_n(t)}{\int t^{1-\nu}(xt)^{\nu}q(tx)dF_n(t)} = \zeta(\infty)$$

since by theorem 4 there exists a $\nu \in [0,1]$ such that $x^{\nu}q(x) \to a \neq 0$, as $x \to \infty$. The composite function $f = \tilde{\zeta}^{-1}\phi$ behaves in a neighborhood of zero as a contractive map, being initially greater than x, then crossing at $\hat{\beta}$ are one being below x for a range, perhaps, thereafter.

Thus we know that successive iterates

$$\beta_{i+1} = f(\beta_i) \quad \text{for } i = 0,1,2 \dots$$

will converge to $\hat{\beta} > 0$ as long as $\beta_0 < \beta^*$, the next larger zero of $\tilde{\zeta} - \phi$, if one exists. Otherwise the iteration will converge to zero.

Moreover for the special cases in (3.2) the inverse $\tilde{\zeta}^{-1}$ can be easily found.

Our computational procedure based on the sample is as follows:

<u>Algorithm</u>: Given $t_1,\ldots,t_k$ as failure times and $t_{k+1},\ldots,t_n$ as censored times from a mixed exponential distribution with prescribed $Q \in \mathcal{Q}$ proceed as follows:

(i) Compute the sample moments $\tilde{t}, \bar{t}, \overline{t^2}$.

(ii) If $\overline{t^2} < 2\bar{t} \cdot \tilde{t}$, assume the observations are from an exponential distribution with failure rate λ and estimate it by

$$\hat{\lambda} = \frac{k}{n\bar{t}} \quad .$$

(iii) If $\overline{t^2} > 2\bar{t}\, \tilde{t}$, assume the observations are from a Mixed Exponential distribution, with prescribed $Q \in \mathcal{Q}$.

Using the sample functions explicitly given by

$$\phi(x) = x \sum_1^n t_i q(xt_i) / \sum_1^n Q(xt_i) \qquad \tilde{\zeta}(x) = 1 + x \sum_1^k [t_i q'(xt_i)/q(xt_i)],$$

we guess β_0 , then iterate using the inductive step;

given β_i, compute $\phi(\beta_i)$ and calculate β_{i+1} such that

$$\tilde{\zeta}(\beta_{i+1}) = \phi(\beta_i).$$

we find $\hat{\beta} = \lim \beta_i$, and compute

$$\hat{\alpha} = k / \sum_1^n Q(t_i\hat{\beta}).$$

The nature of the intersection guarantees that within the region when ϕ and $\tilde{\zeta}$ both decrease the iteration will rapidly converge, with a reasonable first guess. When the functions q and Q are simple a small programmable electronic calculator, such as the HP-67, can be used to obtain these estimates.

We now turn to an examination of the conditions on a sample which will guarantee that it can satisfy (4.4).

Since, remarkably enough, condition (4.4) is the same for any $Q \varepsilon \mathfrak{Q}$ we shall say a sample from a DFR distribution, containing k dead times and $n-k$ alive times, is *ample* whenever it satisfies (4.4); i.e., the following inequality is met,

$$2 \sum_1^k t_i \sum_1^n t_j < k \sum_1^n t_i^2 . \tag{5.1}$$

One can check that a complete sample of failure times, i.e., with $k = n$, will satisfy (5.1) iff the sample standard deviation exceeds the sample mean. Of course, we know that for any DFR distribution the standard deviation does exceed the mean, when they both exist. Thus for most complete samples from Mixed Exponential distributions, (which are sufficiently large), the requirement (5.1) should be satisfied.

Remark: A truncated sample with $k < n$ failure times and the remaining observations truncated at t_0 will satisfy (5.1) if

$$t_0 > \tilde{t} \left(1 + \sqrt{\frac{2k}{n-k} + 1}\right) \cong \tilde{t}\left(\frac{2n-k}{n-k}\right) \text{ for } \frac{k}{n} \text{ small}.$$

To see this note $\Sigma t_i^2 > (n-k)t_0^2$ and so (5.1) must hold if

$$t_0^2 > 2t_0\tilde{t} + \frac{2k}{n-k} (\tilde{t})^2.$$

By the quadratic formula this is equivalent to the inequality stated. The second approximation follows from the first two terms of the binomial expansion.

Some studies of the frequency with which samples from mixed exponentials satisfy condition (5.1) have been made by Sunjata in an unpublished thesis (1974).

6. THE CONFRONTATION WITH DATA

We now present some data sets from two different lots of flight control electronic packages. Each package has recorded, in minutes, either a failure time or an alive time. An alive time is the time the life test was terminated with the package still functioning.

First Data Set

Failure times: 1, 8, 10

Alive times: 59, 72, 76, 113, 117, 124, 145, 149, 153, 182, 320.

Second Data Set

Failure times: 37, 53

Alive times: 60, 64, 66, 70, 72, 96, 123.

One checks that both data sets satisfy condition (5.1) so that both parameters can be estimated in a Mixed Exponential model. We first compute estimates of the failure rate using the total life statistic:

	Data Set 1 $\hat{\lambda} = .00197$	Data Set 2 $\hat{\lambda} = .00312$
time t in min	estimate of reliability at time t, $\hat{R}_1(t)$	estimate of reliability at time t, $\hat{R}_2(t)$
6	.988	.981
10	.980	.969
30	.943	.911
50	.906	.856
100	.821	.732
130	.774	.667

If we assume the data are observations from the mixed exponential distribution of equation (1.2) then using the estimation techniques derived previously in this paper we have the following estimates:

| | Data Set 1 | Data Set 2 |
time t in min	$\hat{\alpha} = .0453$ $\hat{\beta} = 1.03$ estimate of reliability at time t, $\hat{R}_1(t)$	$\hat{\alpha} = .420$ $\hat{\beta} = .01$ estimate of reliability at time t, $\hat{R}_2(t)$
6	.915	.976
10	.896	.961
30	.855	.896
50	.836	.843
100	.810	.747
130	.801	.705
	$R(t) \geq e^{-.047t}$	$R(t) \geq e^{-.0042t}$

Note that these estimates are more consistent with what the data show, that is, for at least the first 50 minutes we expect the reliability estimate for the second set of data to be higher than the reliability estimate for the first set of data. Beyond this time, however, say at 100 minutes, the data indicate that the reliability estimate from the first set of data should be higher than the reliability estimate from the second set of data. Using mixed exponential estimates this is the case.

A statistical test to determine whether the data require a constant or decreasing failure rate was run on the data from sets 1 and 2. For data set 1 we reject constant failure rate (in favor of decreasing failure rate) at the .10 level. For data set 2 we cannot reject the constant failure rate assumption. In this case, however, the constant failure rate estimates for reliability and the mixed exponential estimates for reliability are close.

7. CONCLUSION

If a component has a life distribution with an increasing failure rate, the information necessary to estimate its parameters must contain failure times. In practice this means that if there are few observed failures, within a fleet of operational components, there is little information with which to assess their reliability. If a component has a constant failure rate then both failure times and alive times contribute equally to the estimation of its reliability. This study suggests that if a component has a life distribution with decreasing failure rate it is the alive times within the data which contribute principally to the estimation of the parameters (and thereby to the determination of the reliability) since only one failure observation is required even to estimate two parameters, presuming the data is ample.

The usual justification for using maximum likelihood estimates is due to their asymptotically optimal properties, and to their asymptotic normality. The problem of obtaining the usual sampling distributions of the MLE's of the parameters obtained for Mixed Exponential model seems to be difficult, not only because the estimates are only implicitly defined, but also because samples are virtually always censored. Furthermore, the usual proofs for the asymptotic optimality of the MLE's may not apply when censoring is of a general type and when only sparse failure data is available. A useful asymptotic theory must be developed for censored data sets of which the life of electronic packages of integrated circuits are an illustration.

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